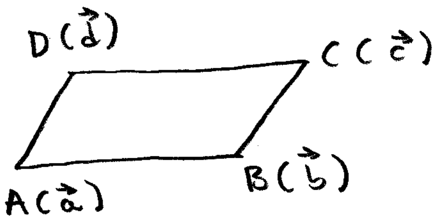
J11 AG
PPS 57-72
VECTOR APPS

[1]



ABCD parallelogram iff $\begin{cases} \vec{AB} \parallel \vec{DC} \\ \text{AND} \\ \vec{AD} \parallel \vec{BC} \end{cases}$

$\vec{AB} \parallel \vec{DC}$ iff $\vec{AB} = m\,\vec{DC}$

$\vec{AD} \parallel \vec{BC}$ iff $\vec{AD} = n\,\vec{BC}$, $m, n \in \mathbb{R}$

<u>conditions</u> $\vec{b} - \vec{a} = m(\vec{c} - \vec{d})$

and

$\vec{d} - \vec{a} = n(\vec{c} - \vec{b})$, $m, n \in \mathbb{R}$

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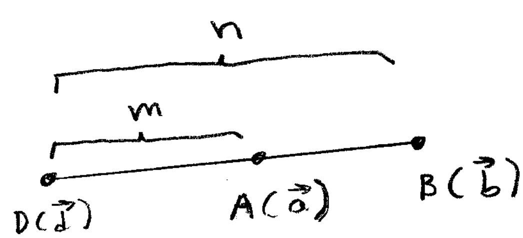
[2]



$$\frac{\vec{DA}}{\vec{DB}} = \frac{m}{n}$$

$$\vec{DA} = \frac{m}{n}\vec{DB}$$

$$\vec{a} - \vec{d} = \frac{m}{n}(\vec{b} - \vec{d})$$

$$n(\vec{a} - \vec{d}) = m(\vec{b} - \vec{d})$$

$$n\vec{a} - n\vec{d} = m\vec{b} - m\vec{d}$$

$$m\vec{d} - n\vec{d} = m\vec{b} - n\vec{a}$$

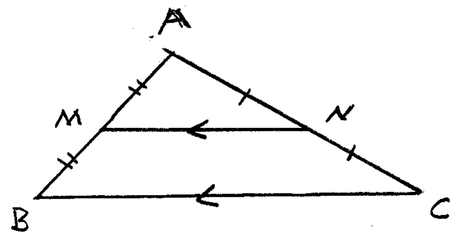
$$\therefore \quad \vec{d} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

□

p 59

[3]

If MN connects midpts of AB and BC, then $MN \parallel BC$ and len $MN = \frac{1}{2}$ len BC.



len $MN = \frac{1}{2}$ len BC

Proof Let $\vec{a}, \vec{b}, \vec{c}, \vec{m}, \vec{n}$ be position vectors of points A, B, C, M, N in ΔABC.

$$\vec{m} = \frac{\vec{a}-\vec{b}}{2} \quad , \vec{n} = \frac{\vec{a}-\vec{c}}{2}$$

$$\begin{aligned}
\overrightarrow{MN} &= \vec{n} - \vec{m} \\
&= \frac{\vec{a}-\vec{c}}{2} - \frac{\vec{a}-\vec{b}}{2} \\
&= \tfrac{1}{2}\left[\vec{a} - \vec{c} - \vec{a} + \vec{b}\right] \\
&= \tfrac{1}{2}\left[-\vec{c} + \vec{b}\right] \\
&= -\tfrac{1}{2}\left[\vec{b} - \vec{c}\right] \\
&= -\tfrac{1}{2}\overrightarrow{CB}
\end{aligned}$$

$$\overrightarrow{MN} = \tfrac{1}{2}\overrightarrow{BC}$$

$\therefore$ MN is $\frac{1}{2}$ len of BC and $MN \parallel BC$.

[4] Thm: Suppose $C(x,y)$ internally divides AB in ratio $m:n$, where $A(x_1,y_1)$, $B(x_2,y_2)$, then

$$x = \frac{mx_2+nx_1}{m+n}, \quad y = \frac{my_2+ny_1}{m+n}.$$

Proof

$$\vec{c} = \frac{m\vec{b}+n\vec{a}}{m+n},$$ Dem 1 p 58, $\vec{a},\vec{b},\vec{c}$ position vectors of A, B, C.

$$\Leftrightarrow \quad \langle x,y\rangle = \frac{m\langle x_2,y_2\rangle + n\langle x_1,y_1\rangle}{m+n}$$

$$= \frac{1}{m+n}\left[\langle mx_2, my_2\rangle + \langle nx_1, ny_1\rangle\right]$$

$$= \frac{1}{m+n}\left[\langle mx_2+nx_1, my_2+ny_1\right]$$

so

$$x = \frac{mx_2+nx_1}{m+n}, \quad y = \frac{my_2+ny_1}{m+n}$$

$\square$

P61

[1.1] ℓ through $(2,-3)$ with direction vector $\langle 1, 2 \rangle$.

$$x = 2 + t$$
$$y = -3 + 2t$$

[1.2] ℓ through $(4,0)$, direction vector $(-3, 2)$

$$x = 4 - 3t$$
$$y = 2t$$

P62

[2.1] $A(-3,2)$, $B(4,5)$

$$\vec{AB} = \langle 4+3, 5-2 \rangle$$
$$= \langle 7, 3 \rangle$$

$$\vec{p} = \langle -3, 2 \rangle + t \langle 7, 3 \rangle$$

[2.2] $A(4,0)$, $B(0,3)$

$$\vec{AB} = \langle 0+4, 3-0 \rangle$$
$$= \langle 4, 3 \rangle$$

$$\vec{p} = \langle 4, 0 \rangle + t \langle 4, 3 \rangle$$

[2.3] $A(4,-3)$, $B(-2,5)$

$$\vec{AB} = \langle -2-4, 5+3 \rangle$$

$$\vec{p} = \langle 4, -3 \rangle + t \langle -6, 8 \rangle$$

[2.4] $A(-\frac{2}{3}, -4)$, $B(2,-4)$ $\Rightarrow$ $\vec{AB} = \langle 2 + \frac{2}{3}, -4+4 \rangle$

$$= \langle \tfrac{8}{3}, 0 \rangle$$

$$\vec{p} = \langle -\tfrac{2}{3}, -4 \rangle + t \langle \tfrac{8}{3}, 0 \rangle$$

P65

[4] $\vec{n} = k < 3, -4 >, \; k \in \mathbb{R}, \; k \neq 0.$

[5] ℓ through $(5, -4)$ $\perp$ $\vec{n} = < 2, 3 >$

$$\ell : \; 2(x-5) + 3(y+4) = 0$$

$$2x - 10 + 3y + 12 = 0$$

$$\boxed{2x + 3y + 2 = 0}$$

P66

[6] $d(P \text{ to } \ell), \; P(2, -5), \; \ell : 4x - 3y + 7 = 0$

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{4(2) + (3)(5) + 7}{\sqrt{16 + 9}}$$

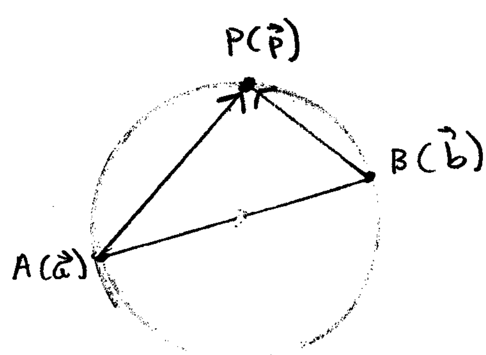
$$\boxed{d = 6}$$

□

P67

[1] $A(\vec{a}), B(\vec{b})$ end pts of diam AB. $P(\vec{p})$ moving pt on circle. Then vector eqn of circle is

$$< \vec{p} - \vec{a} > \cdot < \vec{p} - \vec{b} > = 0$$



Proof

From geometry we know that angle APB is a right angle. So the inner product of $\vec{AP} \cdot \vec{BP} = 0$; i.e.

$$< \vec{p} - \vec{a} > \cdot < \vec{p} - \vec{b} > = 0$$

Hans Eric improved this by noting that wh
P coincides with either A or B,
the dot product is still zero

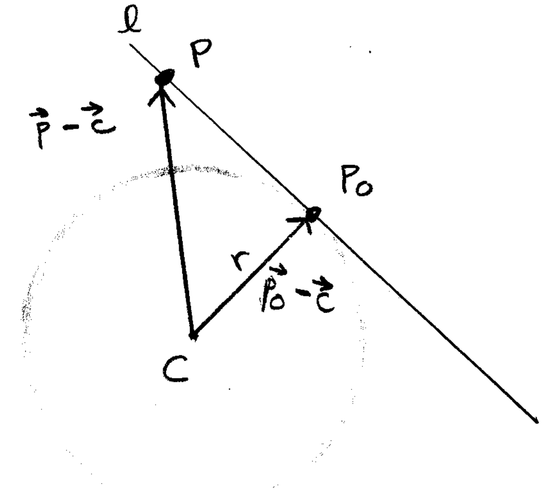
□

P 67

[2] <u>Prove</u> : P_0 on circle of radius r with line ℓ tangent at P_0, show that vector eqn of ℓ is

$$\langle \vec{p} - \vec{c} \rangle \cdot \langle \vec{P_0} - \vec{c} \rangle = r^2$$

where p is movable point on ℓ.



<u>Proof</u>

$$LHS = \langle \vec{p} - \vec{c} \rangle \cdot \langle \vec{P_0} - \vec{c} \rangle$$

$$= |\vec{p} - \vec{c}|\,|\vec{P_0} - \vec{c}| \cos\theta$$

$$= |\vec{p} - \vec{c}|\, r \cos\theta$$

$$= |\vec{p} - \vec{c}|\, r \frac{|\vec{P_0} - \vec{c}|}{|\vec{p} - \vec{c}|}$$

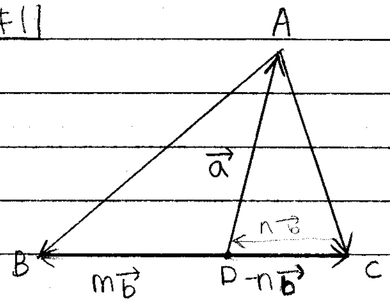
$$= |\vec{P_0} - \vec{c}|\, r$$

$$= r^2$$

$$= R.H.S$$

□

P68 #11



Prove
$$nAB^2 + mAC^2 = nBD^2 + mCD^2 + (m+n)AD^2$$

Proof

$$\begin{aligned}
\text{LHS} &= nAB^2 + mAC^2 \\
&= n(m\vec{b} - \vec{a})^2 + m(-n\vec{b} - \vec{a})^2 \\
&= n(|m\vec{b}|^2 - 2m\vec{a}\cdot\vec{b} + |\vec{a}|^2) + m(|-n\vec{b}|^2 + 2n\vec{a}\cdot\vec{b} + |\vec{a}|^2) \\
&= n|m\vec{b}|^2 - \cancel{2nm\vec{a}\cdot\vec{b}} + n|\vec{a}|^2 + m|-n\vec{b}|^2 + \cancel{2nm\vec{a}\cdot\vec{b}} + m|\vec{a}|^2 \\
&= n|m\vec{b}|^2 + m|+n\vec{b}|^2 + (n+m)|\vec{a}|^2 \\
&= nDB^2 + mDC^2 + (n+m)DA^2 \\
&= nBD^2 + mCD^2 + (n+m)AD^2 = \text{RHS}
\end{aligned}$$

□

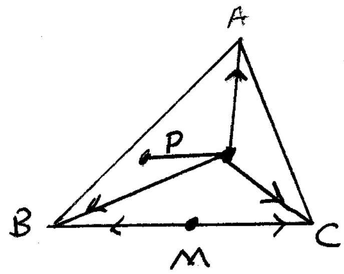
This is DaBeen's proof.

P70

[2] Given:

Suppose H orthocenter of ΔABC. LET P be point such that

$$\overrightarrow{HP} = \frac{1}{2}(\overrightarrow{HA} + \overrightarrow{HB} + \overrightarrow{HC}).$$



Take H as origin of coordinate system so that position vectors of A, B, C, M, P are $\vec{a}, \vec{b}, \vec{c}, \vec{m}, \vec{p}$.

[2.1] Prove: M midpoint of BC $\Longrightarrow$ $\overrightarrow{HA} = 2\overrightarrow{MP}$.

<u>Proof</u>, suppose M midpt of BC.

$$\vec{p} = \frac{1}{2}(\vec{a} + \vec{b} + \vec{c})$$

(1) $2\vec{p} = \vec{a} + \vec{b} + \vec{c}$

$\vec{b} - \vec{m} = -(\vec{c} - \vec{m})$, since M midpt of BC.

$\vec{b} - \vec{m} = \vec{m} - \vec{c}$

(2) $2\vec{m} = \vec{b} + \vec{c}$

1, 2 $\Longrightarrow$ $2\vec{p} = \vec{a} + 2\vec{m}$

$\vec{a} = 2\vec{p} - 2\vec{m}$

$= 2(\vec{p} - \vec{m})$

so that

$\therefore$ $\overrightarrow{HA} = 2\overrightarrow{MP}$

$\square$

$\longrightarrow$

[2.2] Prove: P is the circumcenter of ΔABC. That is, P is the intersection of the perpendicular bisectors of the sides of ΔABC.

<u>Proof</u> since H is orthocenter of ΔABC, $\vec{a}, \vec{b}, \vec{c}$ are perpendicular to the sides of ΔABC. It remains to be shown that M is the midpt of BC. Proofs for the two other midpts is identical to that for M of side BC.

$$\vec{a} \cdot (\vec{b} - \vec{m}) = (\vec{a} \cdot \vec{b}) - (\vec{a} \cdot \vec{m})$$
$$\vec{a} \cdot (\vec{c} - \vec{m}) = (\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{m})$$

So,
$$\vec{a} \cdot (\vec{b} - \vec{m}) - (\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\vec{c} - \vec{m}) - (\vec{a} \cdot \vec{c})$$
$$\vec{a} \cdot (\vec{b} - \vec{m}) - \vec{a} \cdot (\vec{c} - \vec{m}) = (\vec{a} \cdot \vec{b}) - (\vec{a} \cdot \vec{c})$$
$$= \vec{a} \cdot (\vec{b} - \vec{c})$$
$$= 0, \text{ since } \vec{a} \perp \overrightarrow{CB}$$

Then
$$\vec{a} \cdot (\vec{b} - \vec{m}) = \vec{a} \cdot (\vec{c} - \vec{m})$$
$$\vec{b} - \vec{m} = \vec{c} - \vec{m}$$

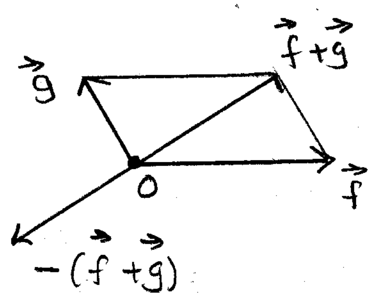
Thus
$$BM = CM$$

Therefore, M midpoint of BC, exactly what we wished to show.

$\square$

P71

[1]

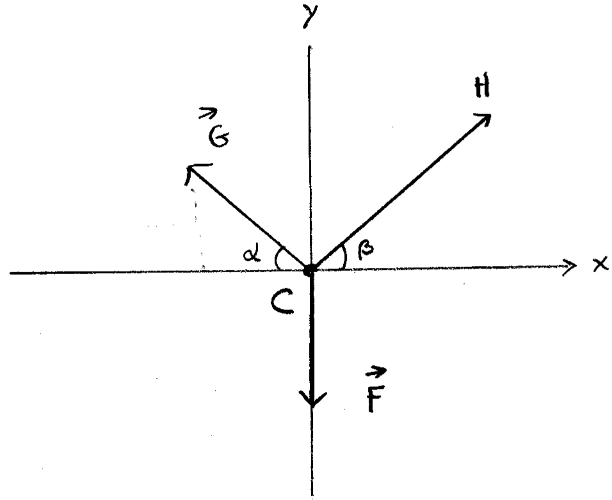


$$\vec{f}+\vec{g}+\vec{h} = 0$$

$$\vec{h} = -(\vec{f}+\vec{g})$$

[2]

No movement $\Rightarrow$ Sum of Forces at C = ZERO.



$$G_x = |\vec{G}| \cos\alpha = \tfrac{3}{5} G$$
$$G_y = |\vec{G}| \sin\alpha = \tfrac{4}{5} G$$
$$H_x = |\vec{H}| \cos\beta = \tfrac{4}{5} H$$
$$H_y = |\vec{H}| \sin\beta = \tfrac{3}{5} H$$

$$\vec{F} = -9.8 \tfrac{m}{s^2} (30\,kg)$$
$$= -294\ N$$

$$\cos\alpha = \tfrac{3}{5} \quad \sin\alpha = \tfrac{4}{5}$$
$$\cos\beta = \tfrac{4}{5} \quad \sin\beta = 3/5$$

$$G_x + H_x = 0$$
$$G_y + H_y = 294\ N$$

$$\begin{bmatrix} -\tfrac{3}{5} G + \tfrac{4}{5} H = 0 \\ \tfrac{4}{5} G + \tfrac{3}{5} H = 294 \end{bmatrix}$$

$$\sim \begin{bmatrix} -3G + 4H = 0 \\ 4G + 3H = 1470 \end{bmatrix}$$

$\Rightarrow G = \quad N, H = \quad N$

$$G_x = 235\ (3/5) = 141\ N$$
$$G_y = 235\ (\tfrac{4}{5}) = 188\ N$$

$$H_x = 176\ (\tfrac{4}{5}) = 141\ N$$
$$H_y = 176\ (\tfrac{3}{5}) = 106\ N$$

$$\therefore \quad G = \langle -141\ N, 188\ N \rangle$$
$$H = \langle 141\ N, 106\ N \rangle$$

P71, ctd

[3] $\vec{V}_0 = \vec{V}_{river} = \langle 2\frac{m}{s}, 0 \rangle$

$\vec{V} = \vec{V}_{BOAT} = \langle 0, 2\frac{m}{s} \rangle$

$$\vec{V}_{actual} = \langle 2\frac{m}{s}, 0 \rangle + \langle 0, 2\frac{m}{s} \rangle$$

$$= \langle 2\frac{m}{s}, 2\frac{m}{s} \rangle$$

$\therefore$ $= 2\sqrt{2} \cdot \frac{m}{s}$ in direction NE

≈ 2.8 m/s NE